▶ **6x6 Stiffness Matrix**

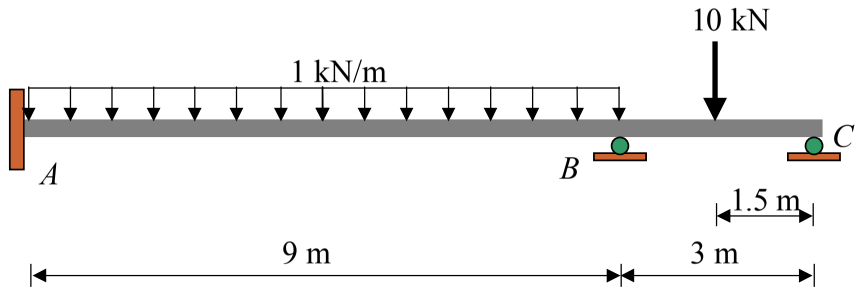
$$[k]_{6\times6} = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

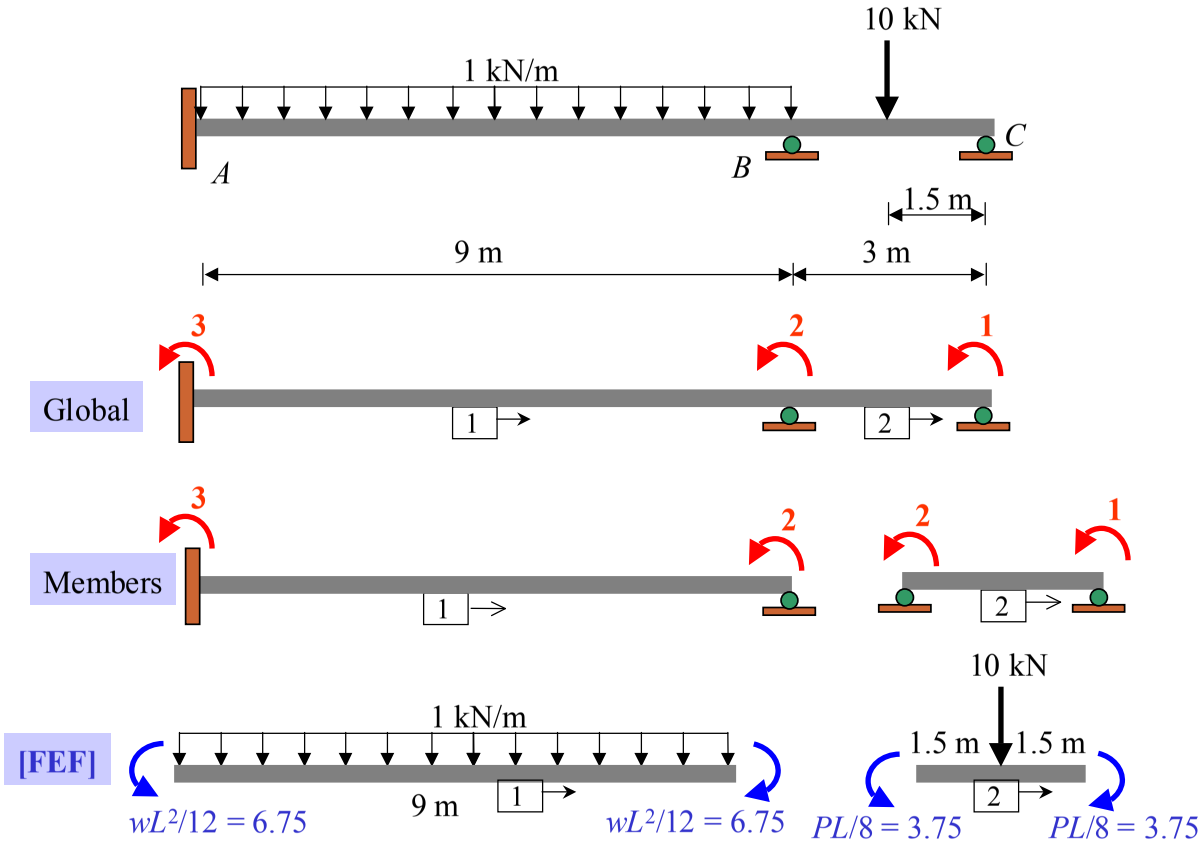
▶ **4x4 Stiffness Matrix**

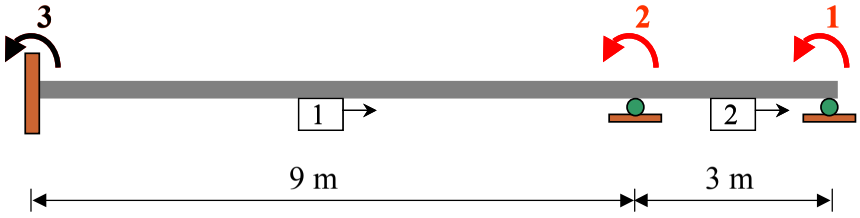
$$[k]_{4\times4} = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

Example 1

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at *B*.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**





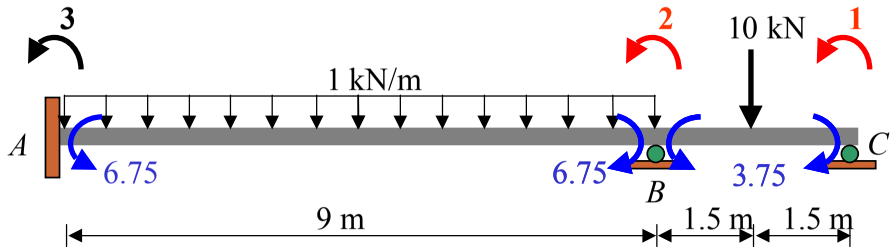


Stiffness Matrix:

$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \begin{matrix} \theta_i & \theta_j \end{matrix} \\ \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix} \end{matrix}$$

$$[k]_1 = EI \begin{matrix} \begin{matrix} 3 & 2 \end{matrix} \\ \begin{bmatrix} 4/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix} \end{matrix} \begin{matrix} 3 \\ 2 \end{matrix} \qquad [k]_2 = EI \begin{matrix} \begin{matrix} 2 & 1 \end{matrix} \\ \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \end{matrix} \begin{matrix} 2 \\ 1 \end{matrix}$$

$$[K] = EI \begin{matrix} \begin{matrix} 2 & 1 \end{matrix} \\ \begin{bmatrix} (4/9)+(4/3) & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \end{matrix} \begin{matrix} 2 \\ 1 \end{matrix}$$



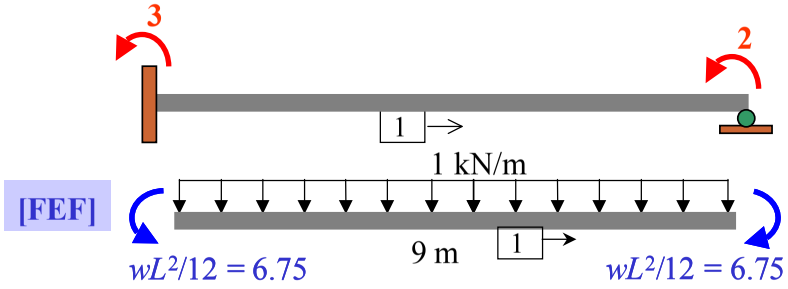
Equilibrium equations: $M_{CB} = 0$

$M_{BA} + M_{BC} = 0$

Global Equilibrium: $[Q] = [K][D] + [Q^F]$

$$\begin{matrix} 2 \\ 1 \end{matrix}\begin{bmatrix} M_{BA} + M_{BC} = 0 \\ M_{CB} = 0 \end{bmatrix} = EI \begin{matrix} \\ 2 \\ 1 \end{matrix}\begin{bmatrix} \overset{2}{(4/9)+(4/3)} & \overset{1}{2/3} \\ 2/3 & 4/3 \end{bmatrix}\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} + \begin{bmatrix} -6.75 + 3.75 = -3 \\ -3.75 \end{bmatrix}$$

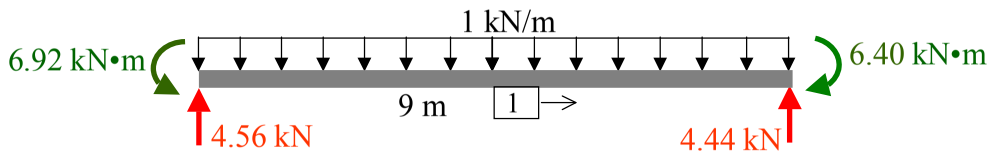
$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0.779/EI \\ 2.423/EI \end{bmatrix}$$

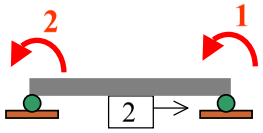


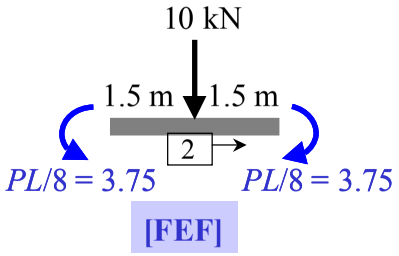
Substitute θ_B and θ_C in the member matrix,

Member 1 **:** $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{matrix} 3 \\ 2 \end{matrix}\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = EI \begin{bmatrix} \overset{3}{4/9} & \overset{2}{2/9} \\ 2/9 & 4/9 \end{bmatrix}\begin{bmatrix} \overset{0}{\theta_A} \\ \theta_B = 0.779/EI \end{bmatrix} + \begin{bmatrix} 6.75 \\ -6.75 \end{bmatrix} = \begin{bmatrix} 6.92 \\ -6.40 \end{bmatrix}$$





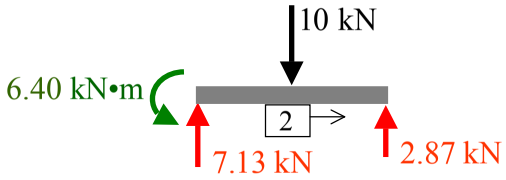


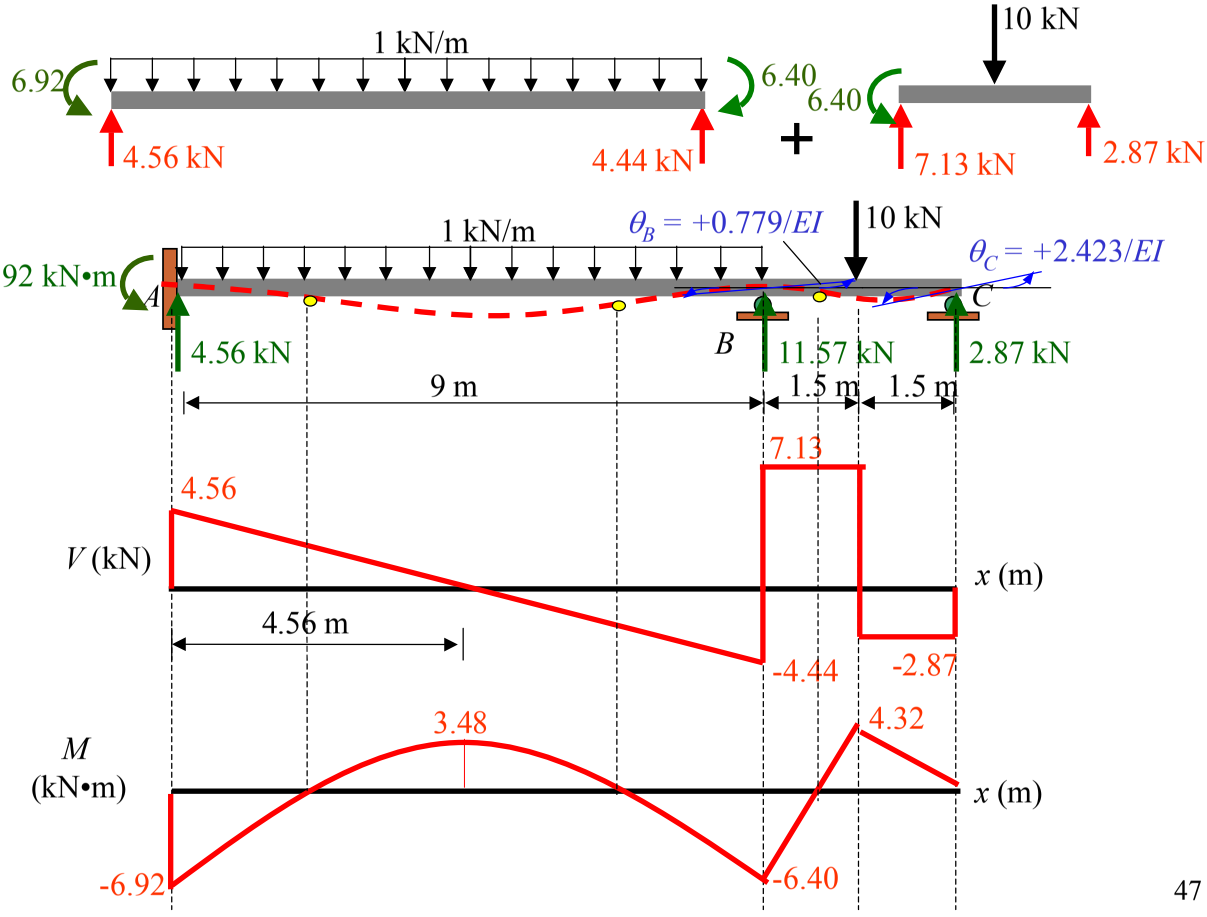
[FEF]

Substitute θ_B and θ_C in the member matrix,

Member 2 : $[q]_2 = [k]_2[d]_2 + [q^F]_2$

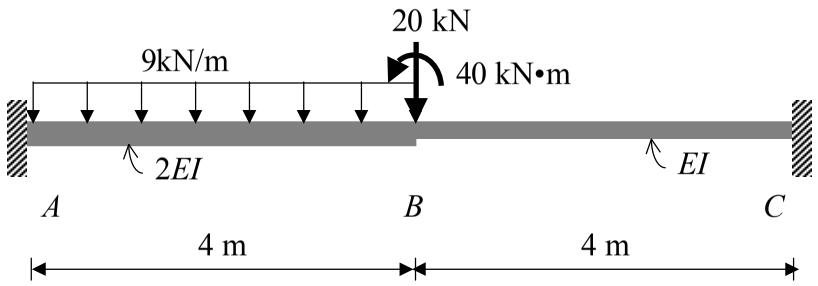
$$\begin{matrix} 2 \\ 1 \end{matrix}\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = EI \begin{matrix} \\ \end{matrix}\begin{matrix} 2 & 1 \end{matrix}\begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix}\begin{bmatrix} \theta_B = 0.779/EI \\ \theta_C = 2.423/EI \end{bmatrix} + \begin{bmatrix} 3.75 \\ -3.75 \end{bmatrix} = \begin{bmatrix} 6.40 \\ 0 \end{bmatrix}$$

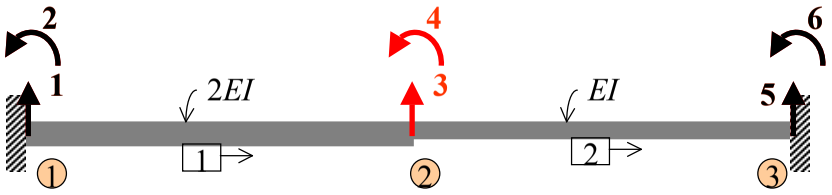




Example 2

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.



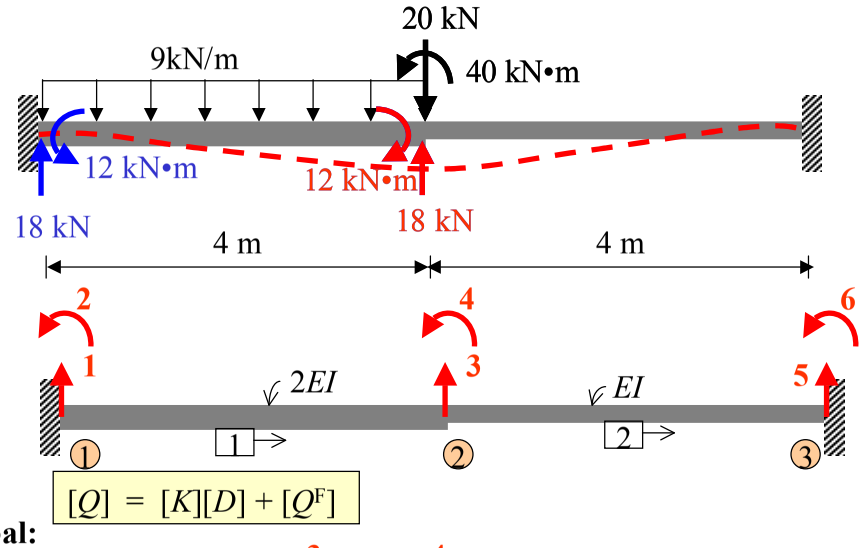


Use 4x4 stiffness matrix,

$$[k] = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array} \qquad [K] = EI \begin{array}{c} \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 0.5625 & -0.375 \\ -0.375 & 3 \end{bmatrix} \end{array}$$

$$[k]_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{bmatrix} \end{array}$$

$$[k]_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{bmatrix} \end{array}$$

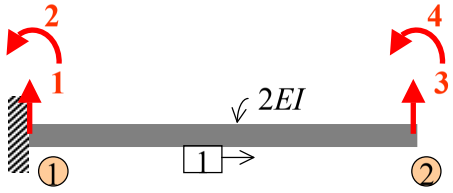


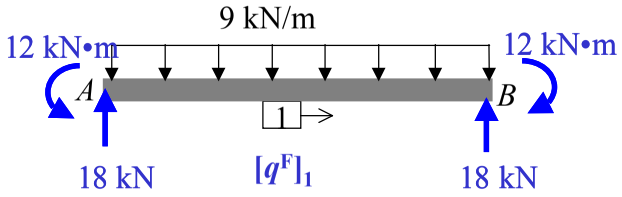
$[Q] = [K][D] + [Q^F]$

Global:

$$\begin{matrix}3\\4\end{matrix}\begin{bmatrix} Q_3 = -20 \\ Q_4 = 40 \end{bmatrix} = EI \begin{matrix}3\\4\end{matrix}\begin{bmatrix} 0.5625 & -0.375 \\ -0.375 & 3 \end{bmatrix} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} + \begin{bmatrix} 18 \\ -12 \end{bmatrix}\begin{matrix}3\\4\end{matrix}$$

$$\begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} -61.09/EI \\ 9.697/EI \end{bmatrix}$$

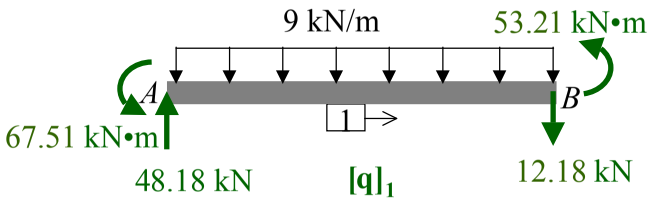


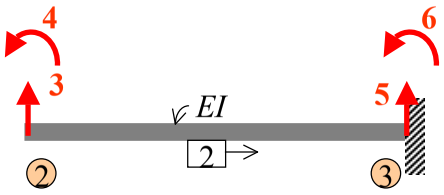


Member 1:

$[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_{3L} \\ q_{4L} \end{bmatrix} = \begin{matrix}1\\2\\3\\4\end{matrix}\begin{bmatrix} 12(2EI)/4^3 & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{bmatrix} \begin{matrix}1&2&3&4\end{matrix} \begin{bmatrix} d_1 = 0 \\ d_2 = 0 \\ d_3 = -61.09/EI \\ d_4 = 9.697/EI \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \\ 18 \\ -12 \end{bmatrix} = \begin{bmatrix} 48.18 \\ 67.51 \\ -12.18 \\ 53.21 \end{bmatrix}$$

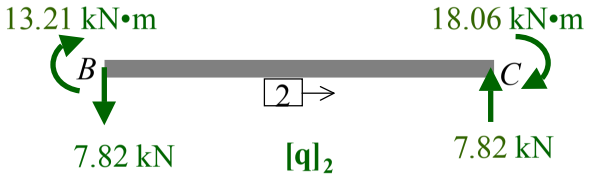


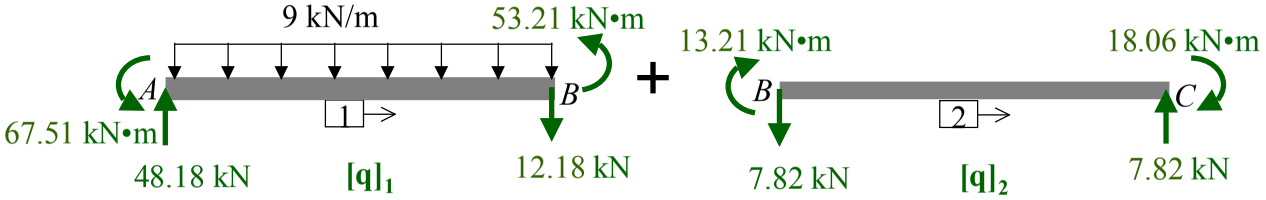


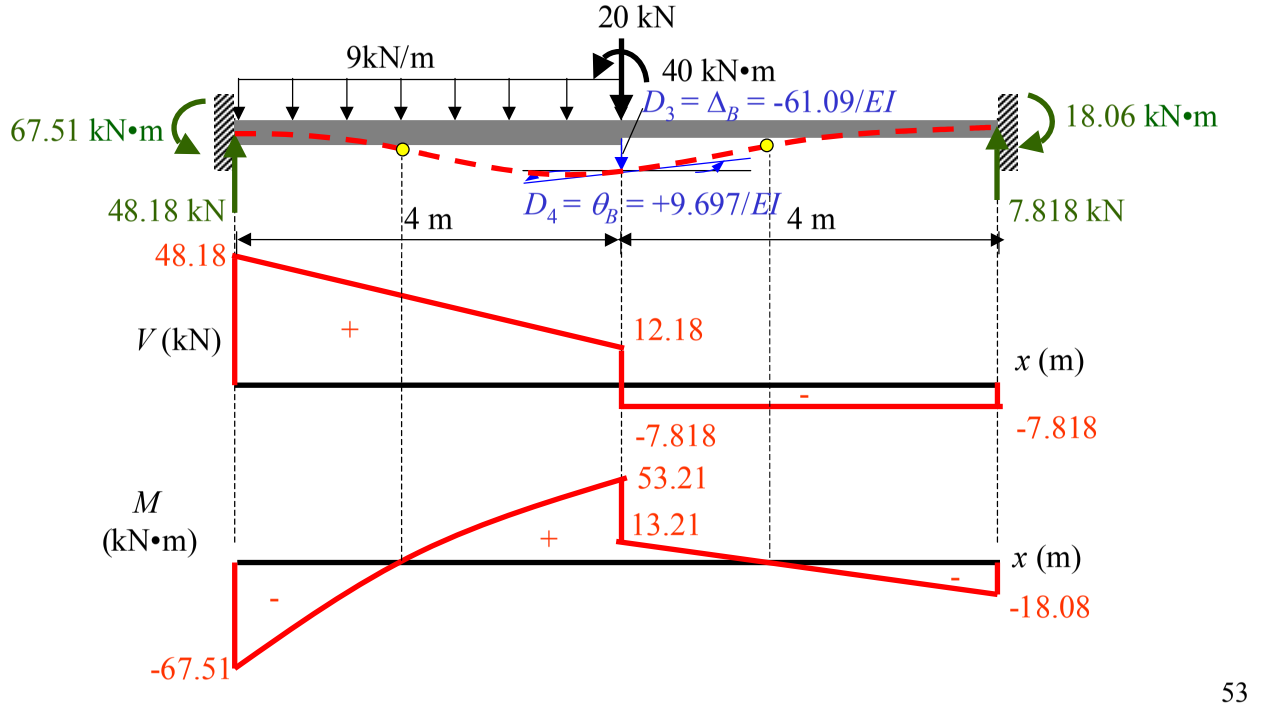
Member 2:

$$[q]_2 = [k]_2[d]_2 + [q^F]_2$$

$$\begin{pmatrix} q_{3R} \\ q_{4R} \\ q_5 \\ q_6 \end{pmatrix} = \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{pmatrix} d_3 = -61.09/EI \\ d_4 = 9.697/EI \\ d_5 = 0 \\ d_6 = 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -7.818 \\ -13.21 \\ 7.818 \\ -18.06 \end{pmatrix}$$

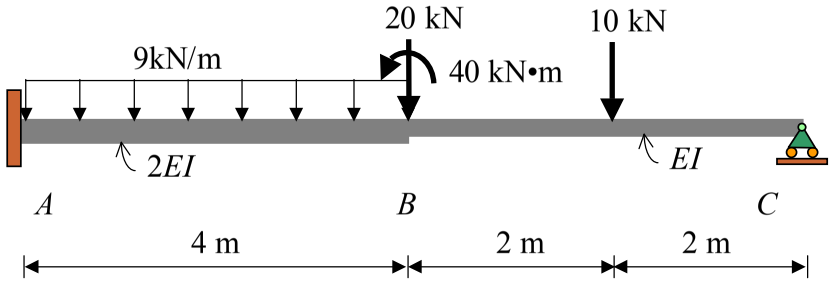


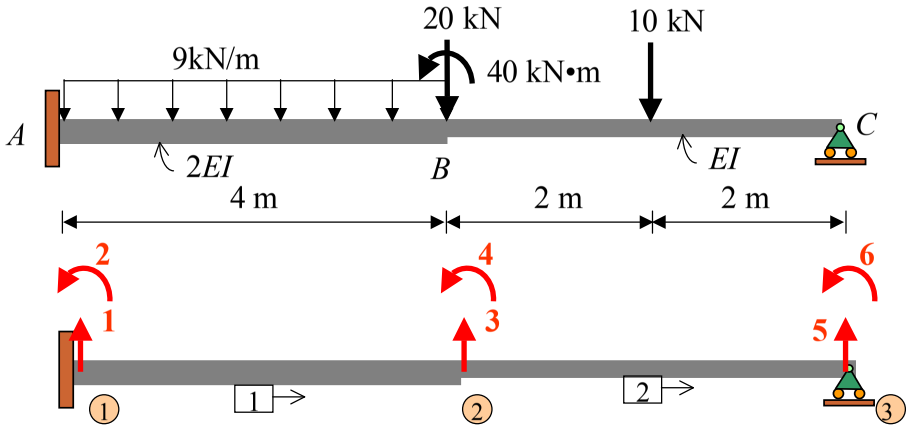




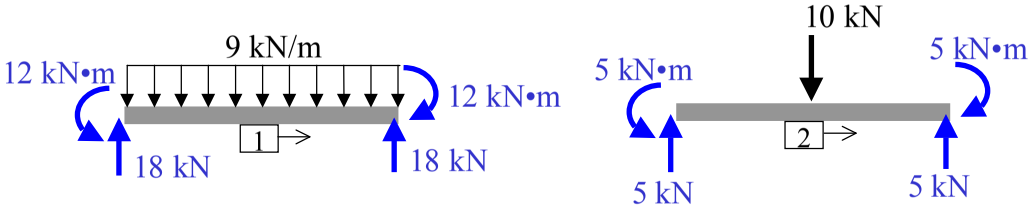
Example 3

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.



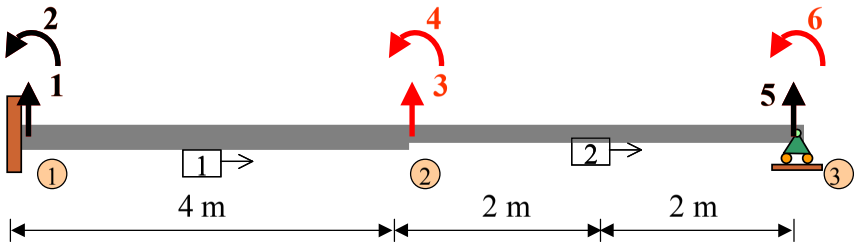


Global



[FEF]

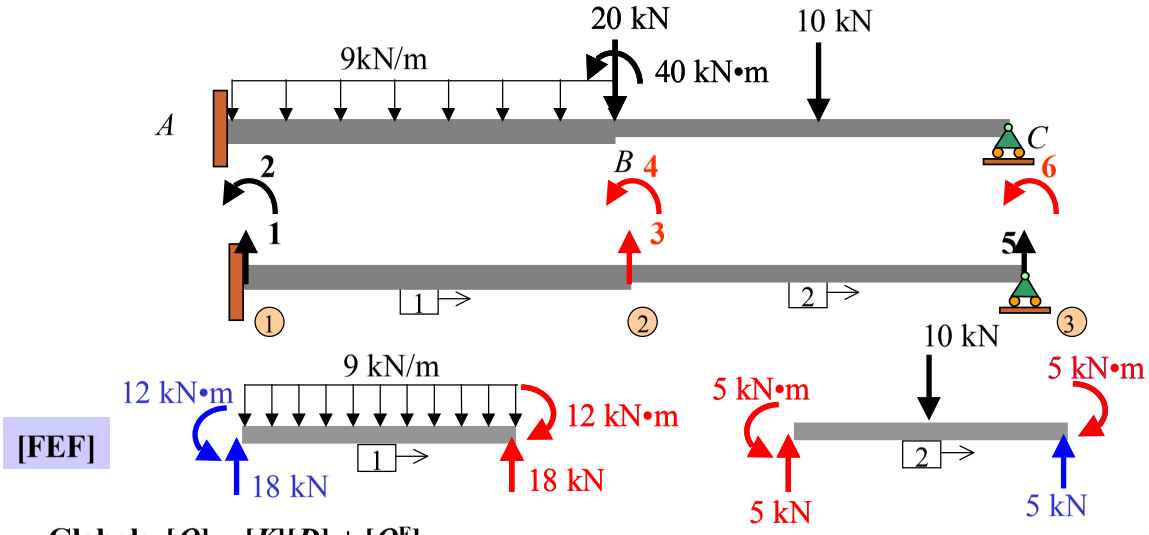
$$[k]_{4\times4} = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$



$$[k]_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12(2EI)/4^3 & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{bmatrix} \end{array}$$

$$[K] = EI \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 0.5625 & -0.375 & 0.375 \\ -0.375 & 3 & 0.5 \\ 0.375 & 0.5 & 1 \end{bmatrix} \end{array}$$

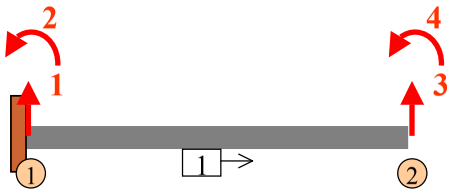
$$[k]_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{bmatrix} \end{array}$$

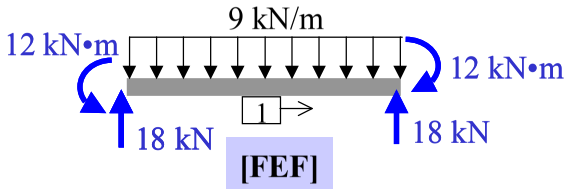


Global: $[Q] = [K][D] + [Q^F]$

$$\begin{bmatrix} V_{BL}+V_{BR} = -20 \\ M_{BA}+M_{BC} = 40 \\ M_{CB} = 0 \end{bmatrix} = EI \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 0.5625 & -0.375 & 0.375 \\ -0.375 & 3 & 0.5 \\ 0.375 & 0.5 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \Delta_B \\ \theta_B \\ \theta_C \end{bmatrix} + \begin{bmatrix} 18+5 = 23 \\ -12+5 = -7 \\ -5 \end{bmatrix}$$

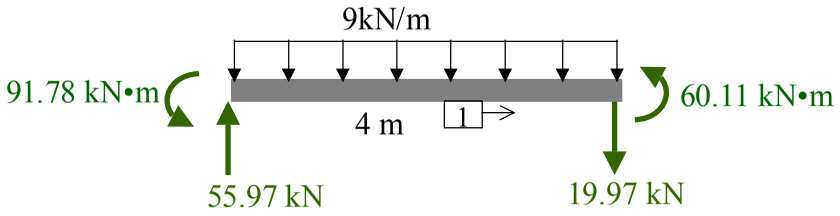
$$\begin{bmatrix} \Delta_B \\ \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} -116.593/EI \\ -7.667/EI \\ 52.556/EI \end{bmatrix}$$

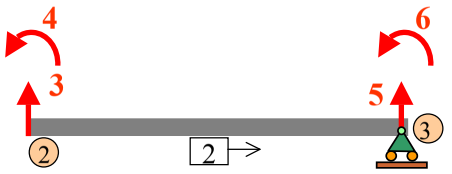


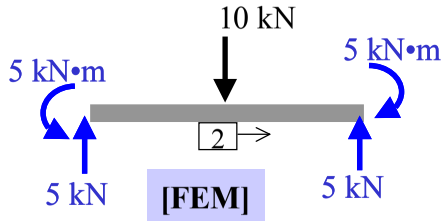


Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} V_A \\ M_{AB} \\ V_{BL} \\ M_{BA} \end{pmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{pmatrix} \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \Delta_B = -116.593/EI \\ \theta_B = -7.667/EI \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 55.97 \\ 91.78 \\ -19.97 \\ 60.11 \end{pmatrix}$$

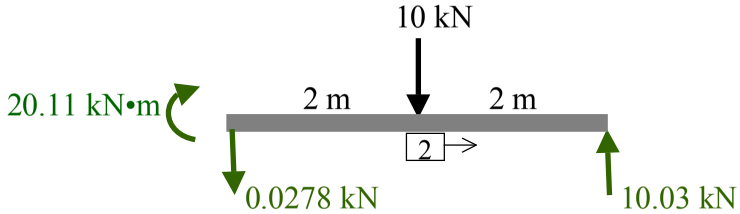


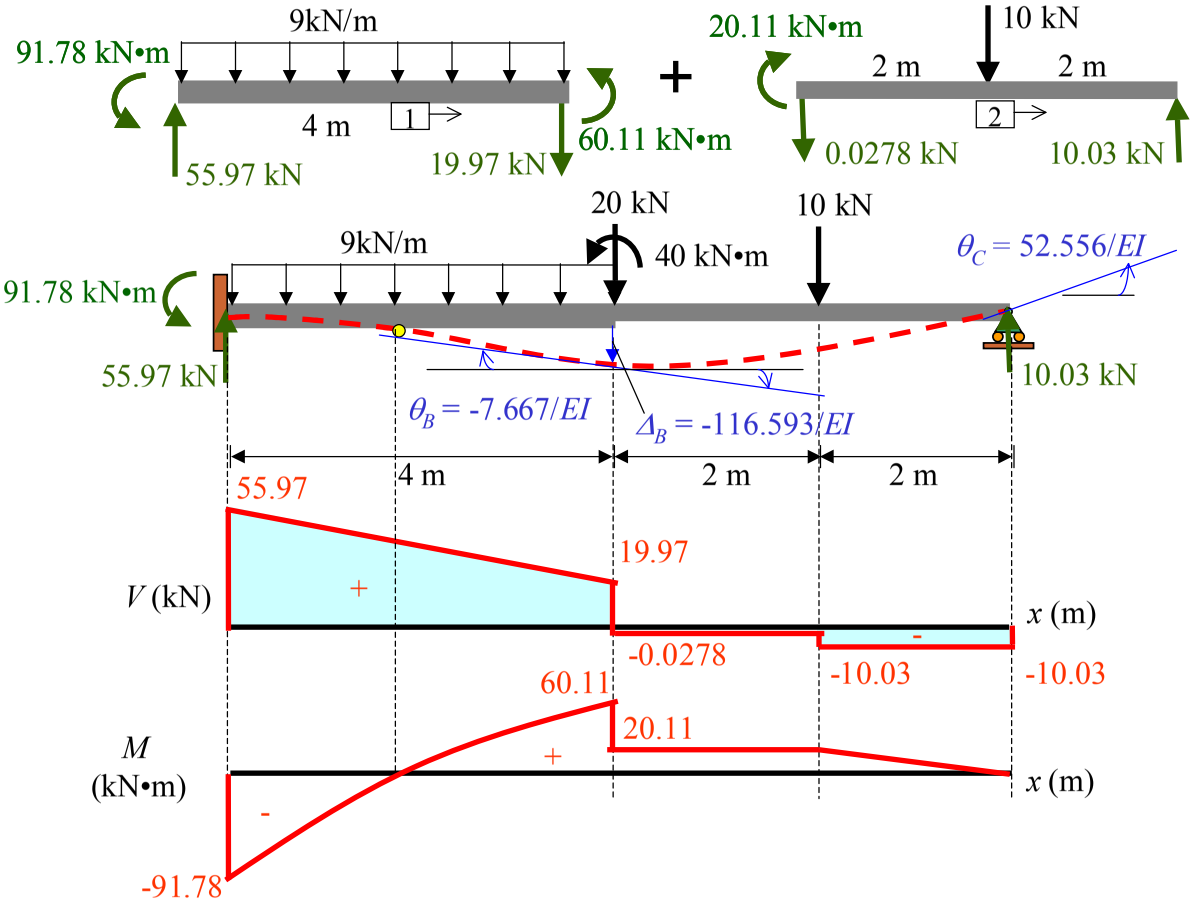




Member 2: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} V_{BR} \\ M_{BC} \\ V_C \\ M_{CB} \end{pmatrix} = \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \end{matrix} \begin{pmatrix} \Delta_B = -116.593/EI \\ \theta_B = -7.667/EI \\ 0 \\ \theta_C = 52.556/EI \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} -0.0278 \\ -20.11 \\ 10.03 \\ 0 \end{pmatrix}$$

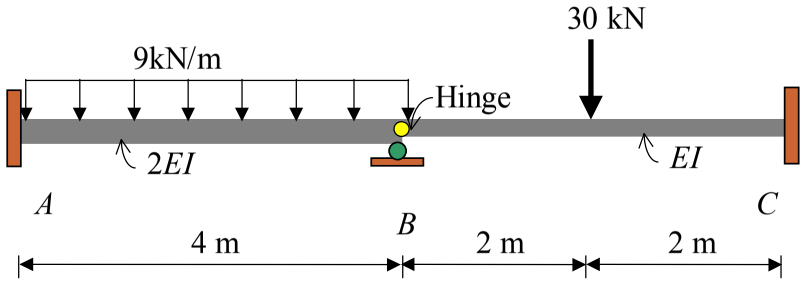


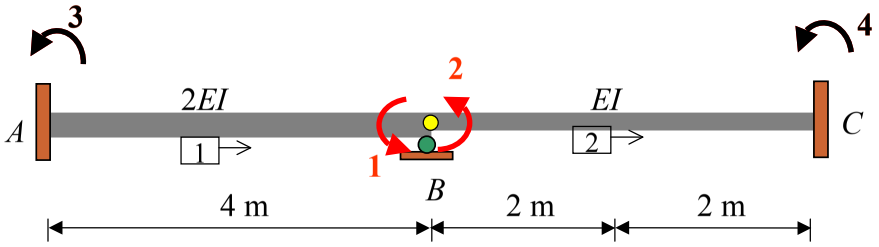


Internal Hinges

Example 6

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
E = 200 GPa, I = 50x10^{-6} m^4.

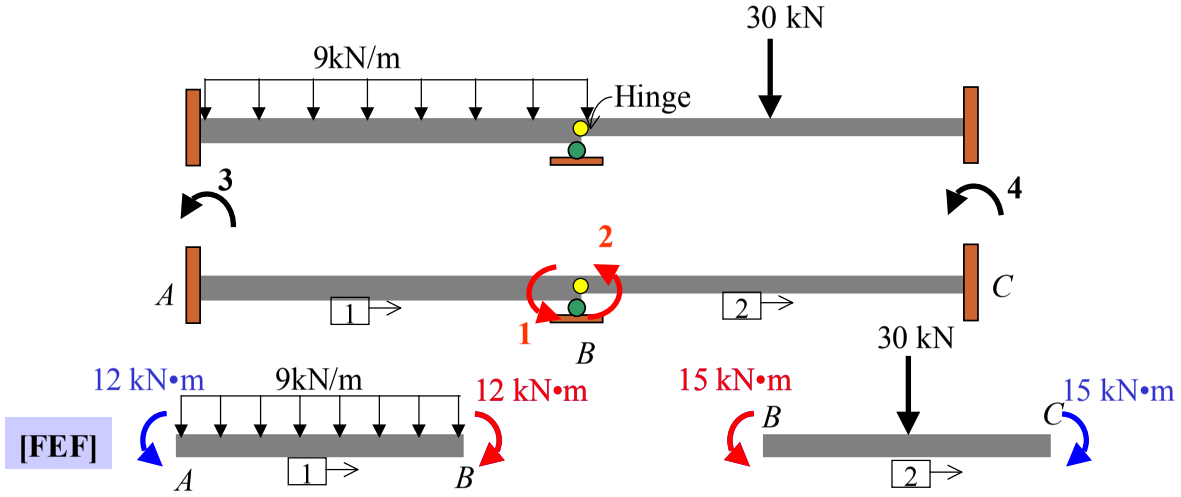




Use 2x2 stiffness matrix, $[k]_{2\times2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \theta_i & \theta_j \\ \end{matrix} \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix}$

$$[k]_1 = \begin{matrix} \\ 3 \\ 1 \end{matrix} \begin{matrix} 3 & 1 \\ \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \qquad [k]_2 = \begin{matrix} \\ 2 \\ 4 \end{matrix} \begin{matrix} 2 & 4 \\ \end{matrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix}$$

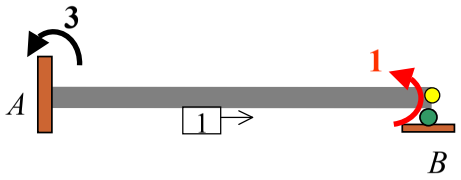
$$[K] = EI \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 & 2 \\ \end{matrix} \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

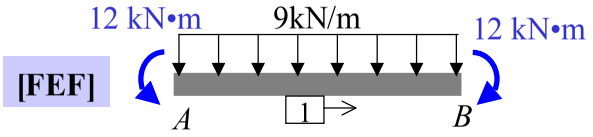


Global matrix:

$$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} = EI \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 & 2 \\ \end{matrix} \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + \begin{bmatrix} -12 \\ 15 \end{bmatrix}$$

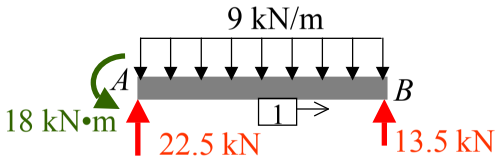
$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0.0006 & \text{rad} \\ -0.0015 & \text{rad} \end{bmatrix}$$

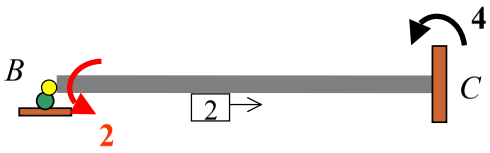


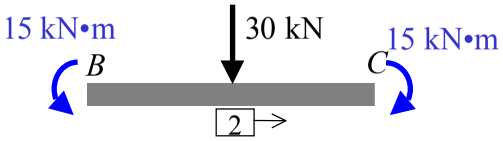


Member 1:

$$\begin{bmatrix} q_3 \\ q_1 \end{bmatrix} = \begin{matrix} 3 \\ 1 \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \begin{bmatrix} d_3 = 0.0 \\ d_1 = 0.0006 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 18 \\ 0.0 \end{bmatrix}$$

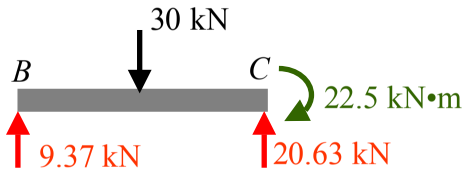


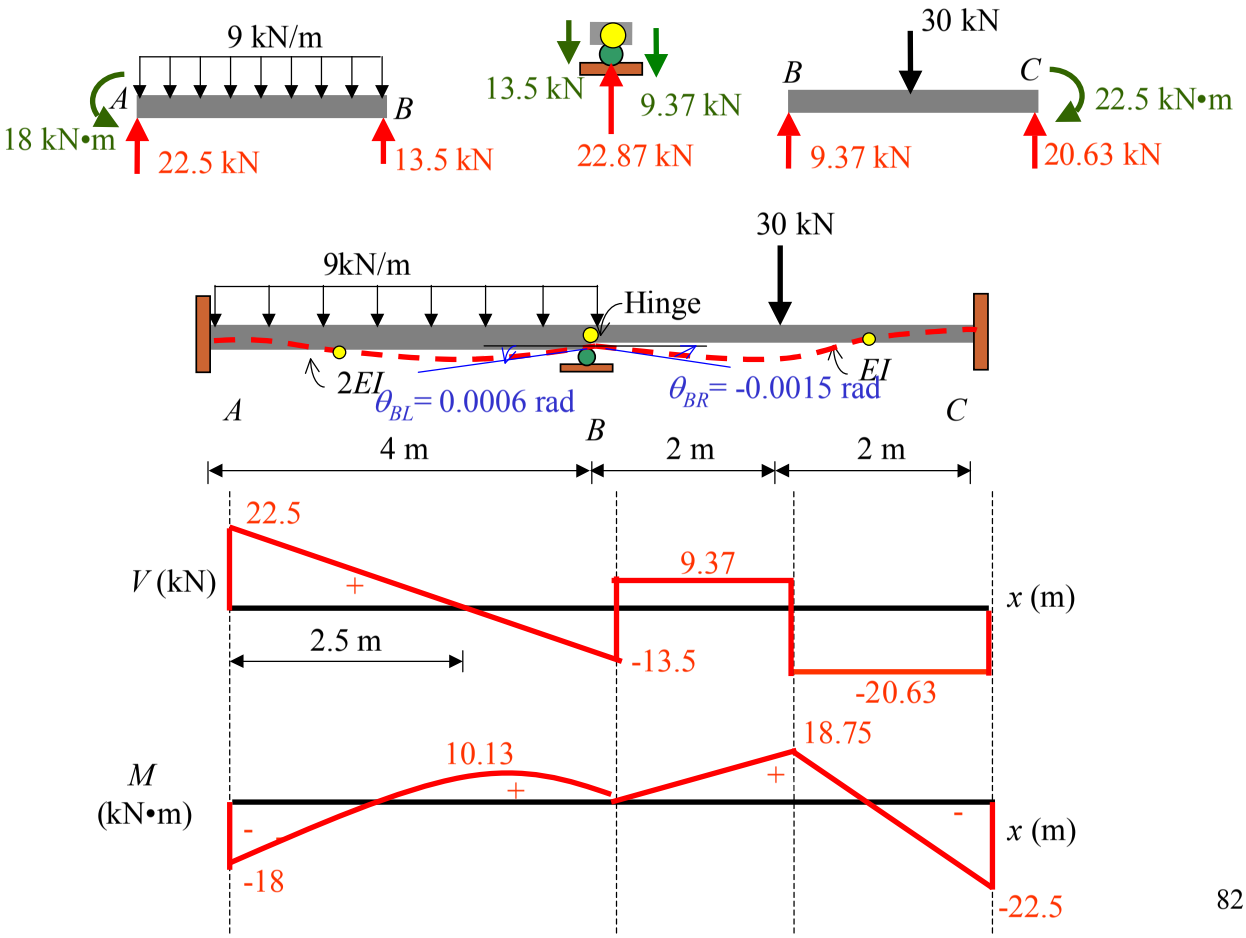




Member 2:

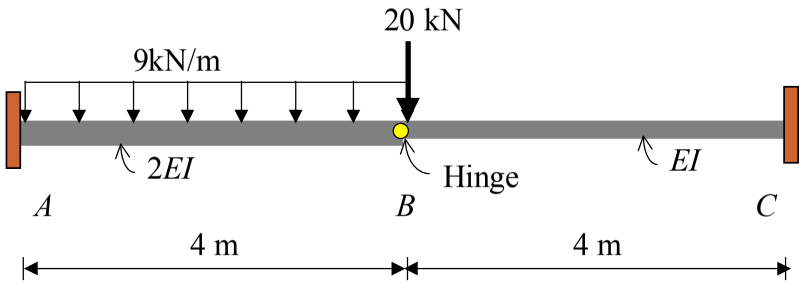
$$\begin{bmatrix} q_2 \\ q_4 \end{bmatrix} = \begin{matrix} 2 \\ 4 \end{matrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix} \begin{bmatrix} d_2 = -0.0015 \\ d_4 = 0.0 \end{bmatrix} + \begin{bmatrix} 15 \\ -15 \end{bmatrix} = \begin{bmatrix} 0.0 \\ -22.5 \end{bmatrix}$$

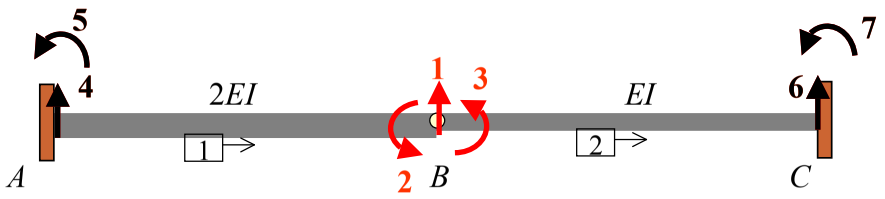




Example 7

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
E = 200 GPa, I = 50x10^{-6} m^4.

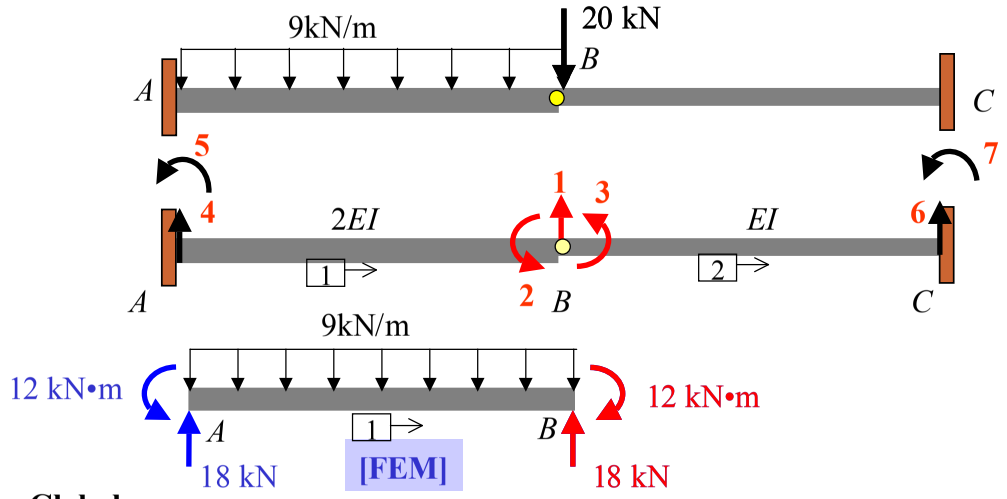




$$[k]_1 = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{bmatrix} \end{array}$$

$$[K] = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array}$$

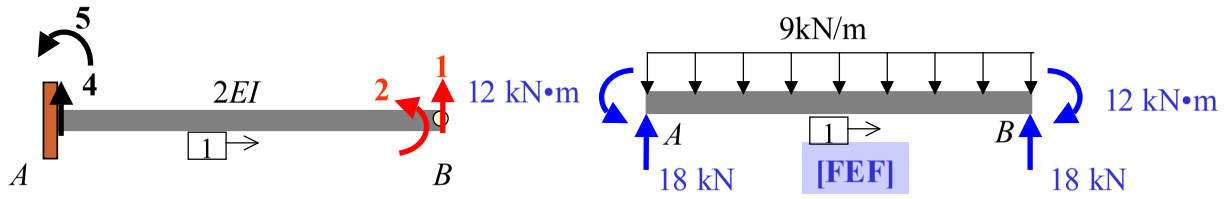
$$[k]_2 = \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{bmatrix} \end{array}$$



Global:

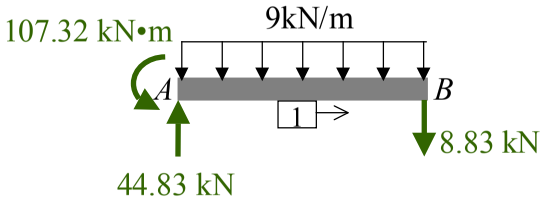
$$\begin{bmatrix} Q_1 = -20 \\ Q_2 = 0.0 \\ Q_3 = 0.0 \end{bmatrix} = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} + \begin{bmatrix} 18 \\ -12 \\ 0.0 \end{bmatrix}$$

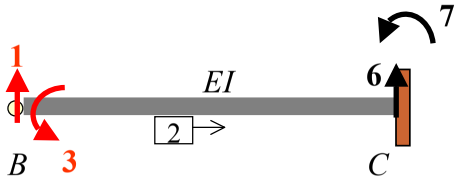
$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} -0.02382 \text{ m} \\ -0.008333 \text{ rad} \\ 0.008933 \text{ rad} \end{bmatrix}$$



Member 1:

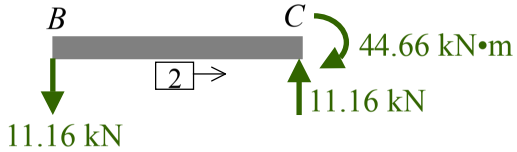
$$\begin{pmatrix} q_4 \\ q_5 \\ q_1 \\ q_2 \end{pmatrix} = \begin{matrix} 4 \\ 5 \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 4 & 5 & 1 & 2 \end{matrix} \\ \begin{pmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{pmatrix} \end{matrix} \begin{pmatrix} d_4 = 0.0 \\ d_5 = 0.0 \\ d_1 = -0.02382 \\ d_2 = -0.00833 \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 44.83 \\ 107.32 \\ -8.83 \\ 0.0 \end{pmatrix}$$

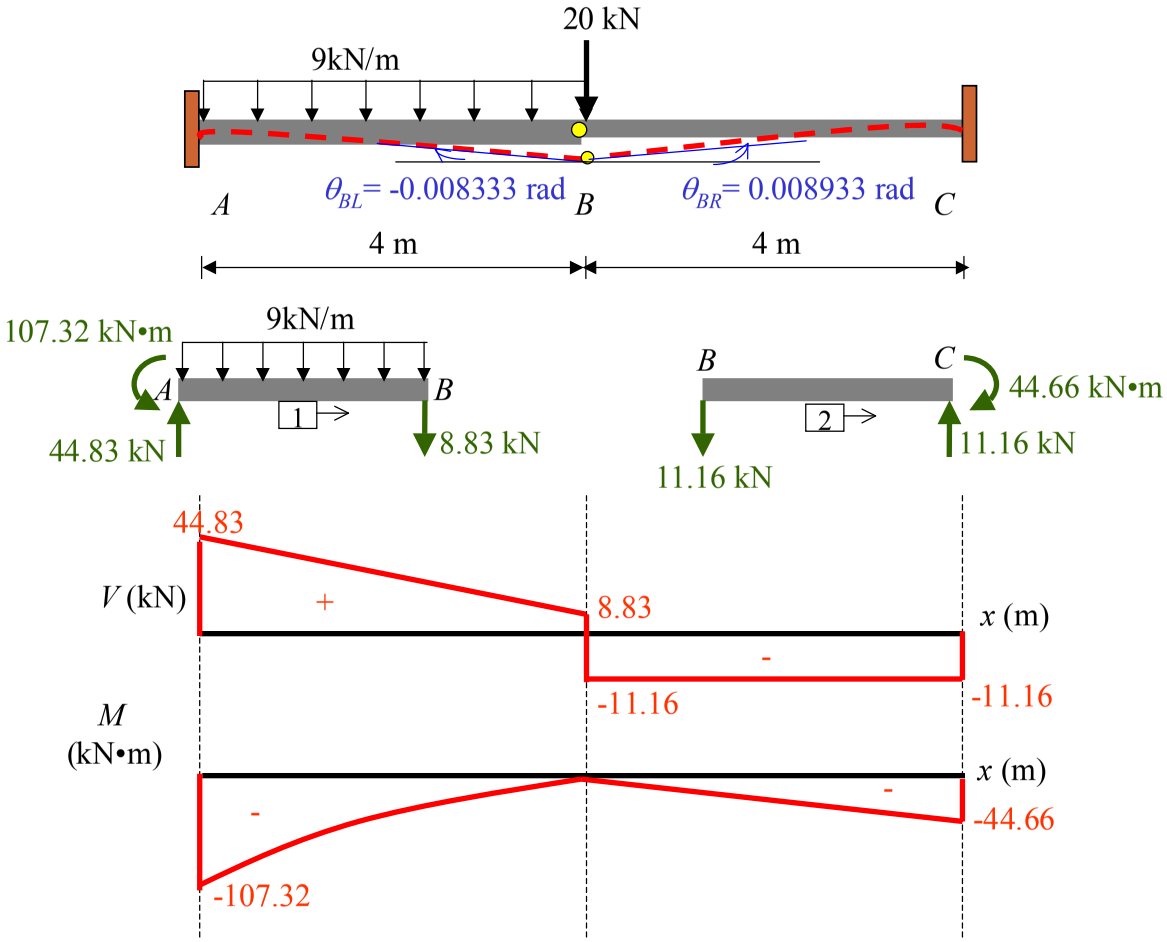




Member 2:

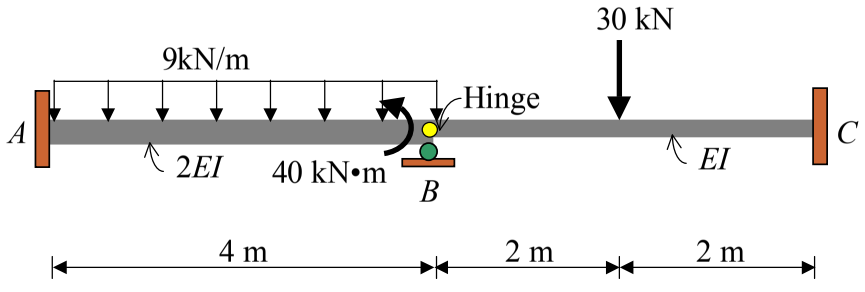
$$\begin{pmatrix} q_1 \\ q_3 \\ q_6 \\ q_7 \end{pmatrix} = \begin{matrix} 1 \\ 3 \\ 6 \\ 7 \end{matrix} \begin{matrix} \begin{matrix} 1 & 3 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \end{matrix} \begin{pmatrix} d_1 = -0.02382 \\ d_3 = 0.008933 \\ d_6 = 0.0 \\ d_7 = 0.0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -11.16 \\ 0.0 \\ 11.16 \\ -44.66 \end{pmatrix}$$

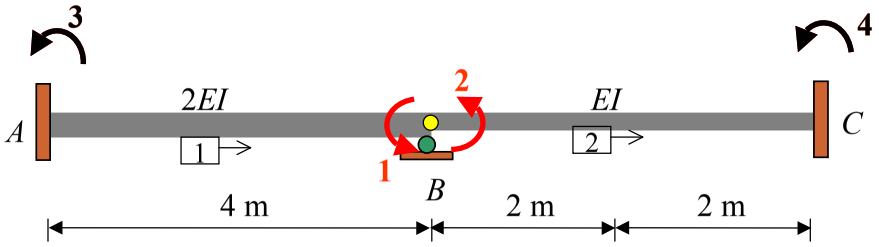




Example 8

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
40 kN•m at the end of member *AB*. E = 200 GPa, I = 50x10^{-6} m^4.



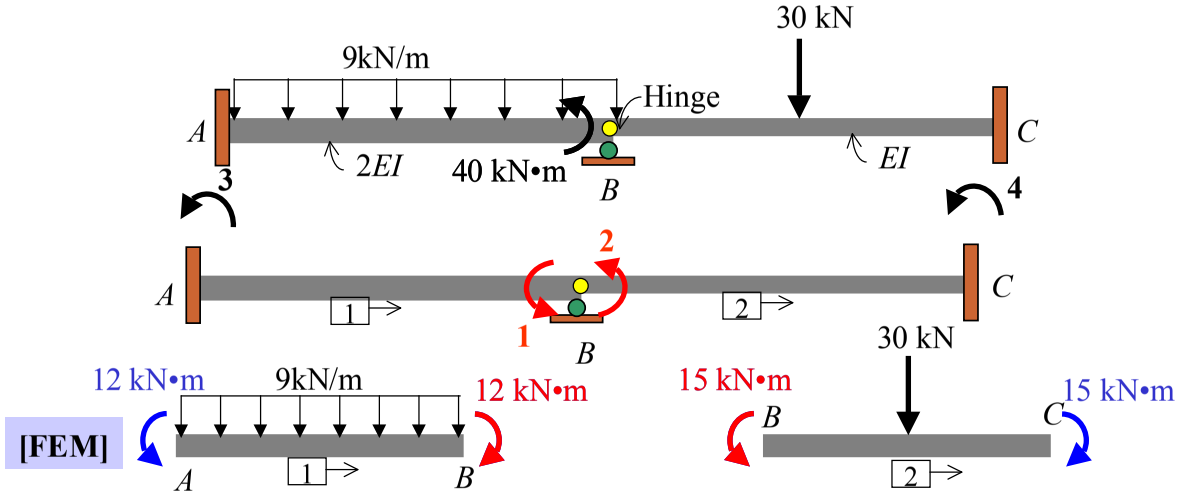


Use 2x2 stiffness matrix:

$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \theta_i & \theta_j \\ \end{matrix} \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix}$$

$$[k]_1 = \begin{matrix} \\ 3 \\ 1 \end{matrix} \begin{matrix} 3 & 1 \\ \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \qquad [k]_2 = \begin{matrix} \\ 2 \\ 4 \end{matrix} \begin{matrix} 2 & 4 \\ \end{matrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix}$$

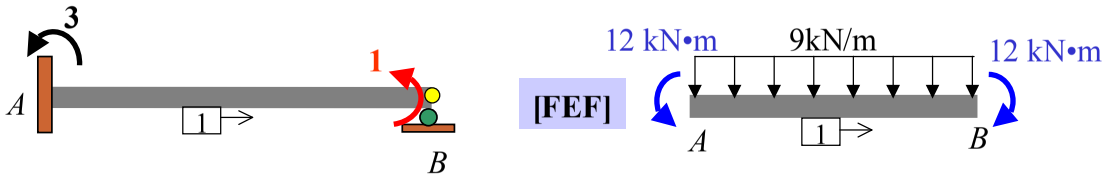
$$[K] = EI \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 & 2 \\ \end{matrix} \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$



Global matrix:

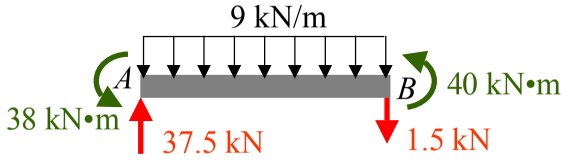
$$\begin{bmatrix} Q_1 = 40 \\ Q_2 = 0.0 \end{bmatrix} = EI \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 & 2 \\ \end{matrix} \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + \begin{bmatrix} -12 \\ 15 \end{bmatrix}$$

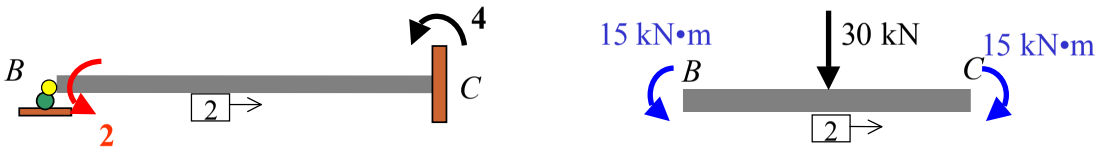
$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0.0026 & \text{rad} \\ -0.0015 & \text{rad} \end{bmatrix}$$



Member 1:

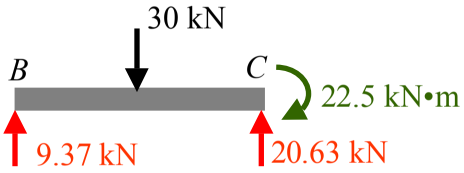
$$\begin{bmatrix} q_3 \\ q_1 \end{bmatrix} = \begin{matrix} 3 \\ 1 \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \begin{bmatrix} d_3 = 0.0 \\ d_1 = 0.0026 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 38 \\ 40 \end{bmatrix}$$

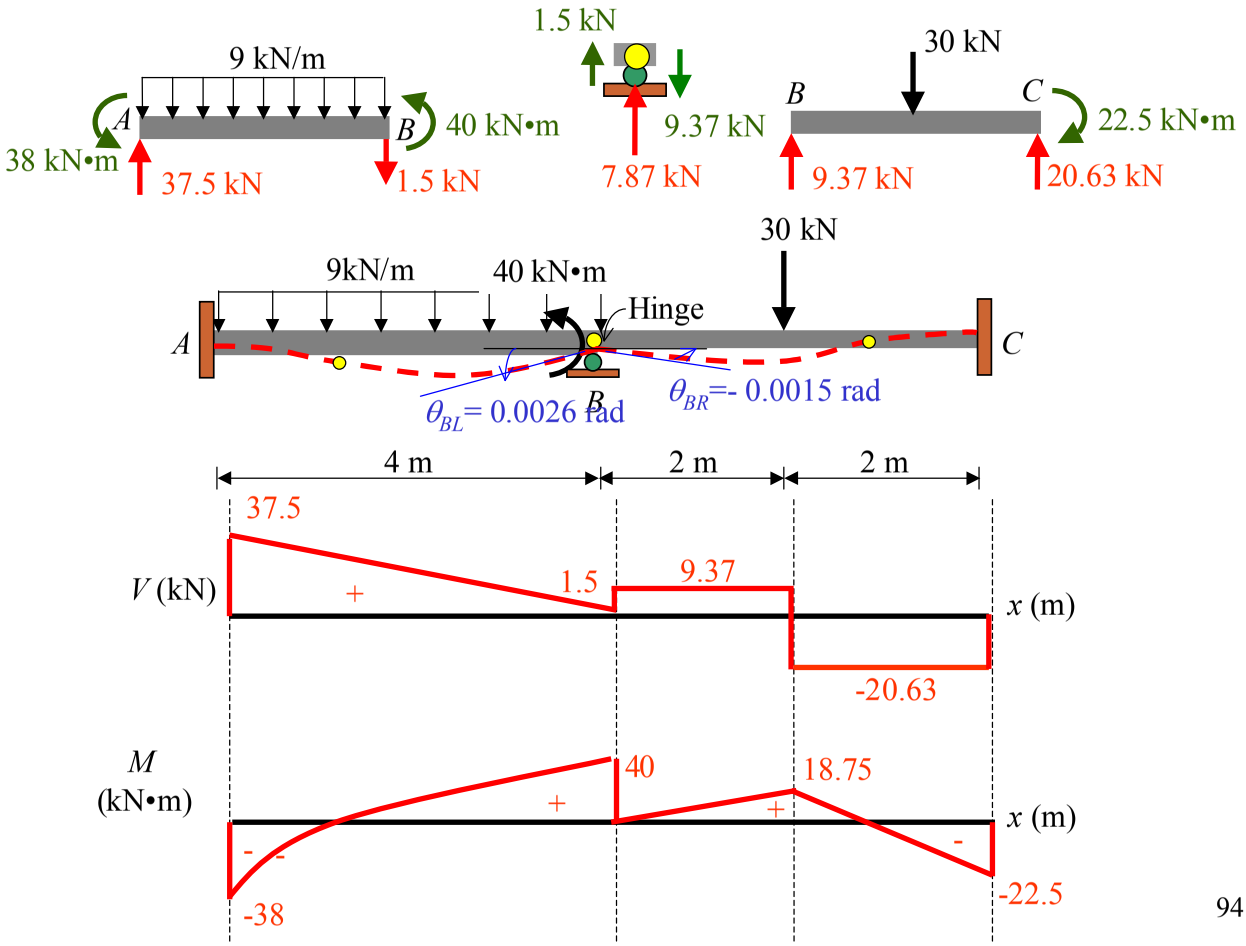




Member 2:

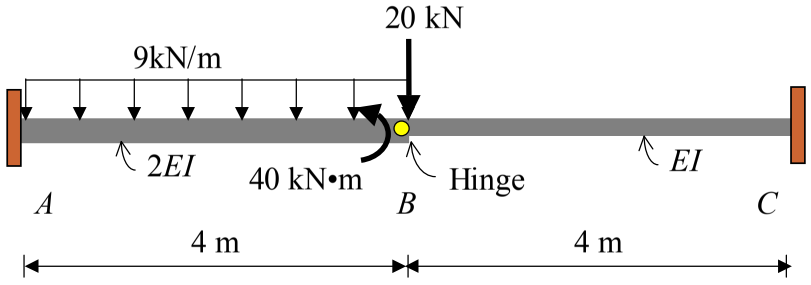
$$\begin{bmatrix} q_2 \\ q_4 \end{bmatrix} = \begin{matrix} 2 \\ 4 \end{matrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix} \begin{bmatrix} d_2 = -0.0015 \\ d_4 = 0.0 \end{bmatrix} + \begin{bmatrix} 15 \\ -15 \end{bmatrix} = \begin{bmatrix} 0.0 \\ -22.5 \end{bmatrix}$$

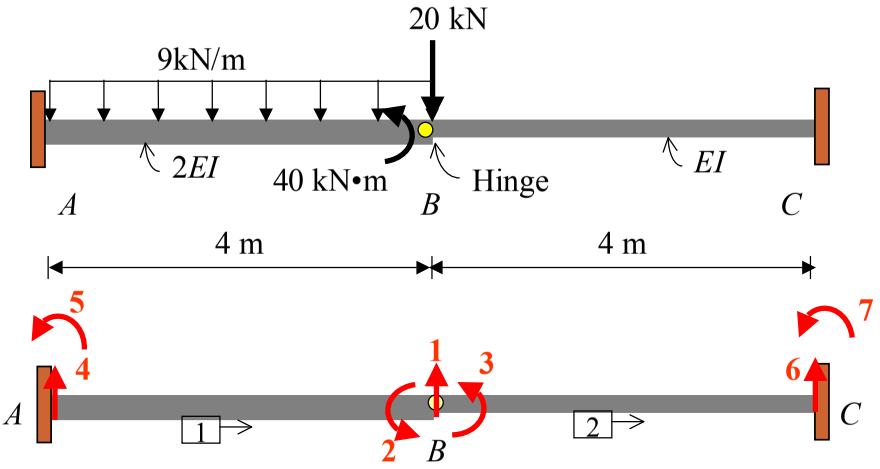


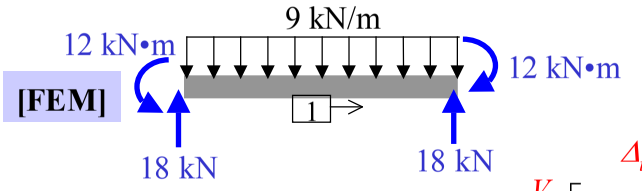


Example 9

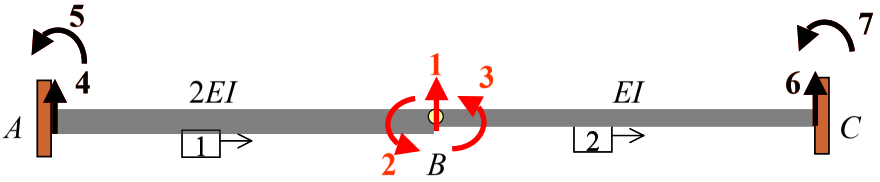
For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
40 kN•m at the end of member *AB*. $E = 200$ GPa, $I = 50\text{x}10^{-6}$ m^4







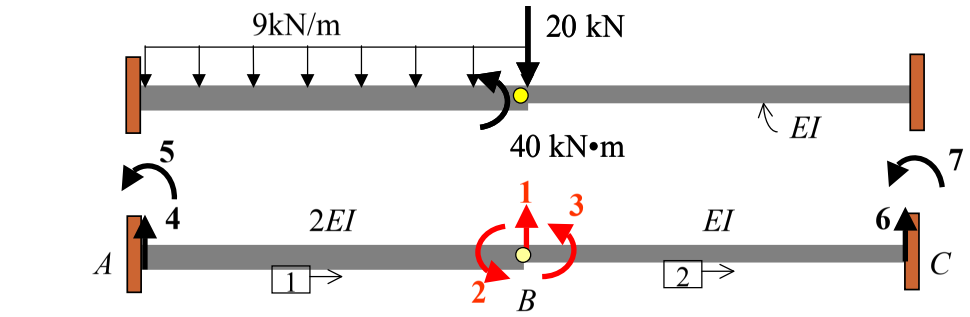
$$[k]_{4\times4} = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

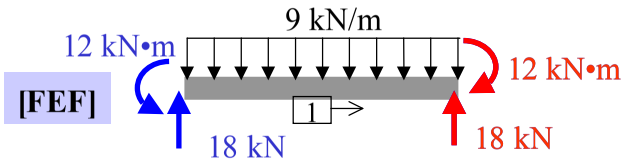


$$[k]_1 = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{bmatrix} \end{array}$$

$$[K] = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array}$$

$$[k]_2 = \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{bmatrix} \end{array}$$

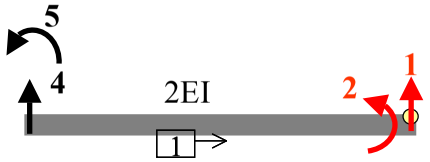


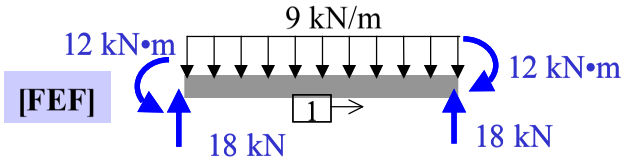


Global:

$$\begin{pmatrix} Q_1 = -20 \\ Q_2 = 40 \\ Q_3 = 0.0 \end{pmatrix} = EI \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \overset{\begin{matrix} 1 & 2 & 3 \end{matrix}}{\begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix}} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} 18 \\ -12 \\ 0.0 \end{pmatrix}$$

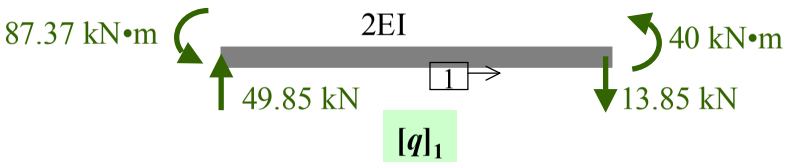
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} -0.01316 \text{ m} \\ -0.002333 \text{ rad} \\ 0.0049333 \text{ rad} \end{pmatrix}$$

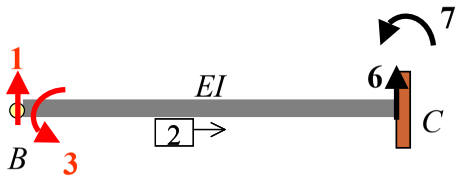




Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_1 \\ q_2 \end{pmatrix} = \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \overset{\begin{matrix} 4 & 5 & 1 & 2 \end{matrix}}{\begin{bmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{bmatrix}} \begin{pmatrix} d_4 = 0.0 \\ d_5 = 0.0 \\ d_1 = -0.01316 \\ d_2 = -0.002333 \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 49.85 \\ 87.37 \\ -13.85 \\ 40 \end{pmatrix}$$





Member 2: $[q]_2 = [k]_2[d]_2 + [q^F]_2$

$$\begin{pmatrix} q_1 \\ q_3 \\ q_6 \\ q_7 \end{pmatrix} = \begin{matrix} 1 \\ 3 \\ 6 \\ 7 \end{matrix} \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \begin{matrix} 1 \\ 3 \\ 6 \\ 7 \end{matrix} \begin{pmatrix} d_1 = -0.01316 \\ d_3 = 0.004933 \\ d_6 = 0.0 \\ d_7 = 0.0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6.18 \\ 0.0 \\ 6.18 \\ -24.69 \end{pmatrix}$$